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Instanton constraints in supersymmetric gauge theories I. Supersymmetric QCD

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Abstract

A previous analysis of possible constraints of Yang-Mills instantons in the presence of spontaneous symmetry breaking is extended to supersymmetric QCD. It is again found that a constraint is necessary for the gauge field in second and fourth order of the gauge breaking parameter v. While the supersymmetric zero mode is well behaved to all orders, the lifted superconformal and quark zero modes show nonpermissible behaviour, but only at first order in v.
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1 Introduction

Constrained instantons [1] have played an important part in considerations on nonperturbative effects in quantum theories, notably in supersymmetric theories. It was, however, realized some time ago [2] that the class of allowed constraints is more restricted than originally envisaged since a scale-fixing constraint is a necessary but not a sufficient condition for a finite action. This makes it interesting to examine the original applications of constrained instantons in the more restricted class of constraints allowed according to [2].

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In the present paper instanton constraints in supersymmetric quantum chromodynamics (SQCD) as originally considered in [3] are investigated from this viewpoint. For the gauge field, where a brief version of the argument of [2] is given for the sake of clarity and completeness, it is again found that a constraint is necessary in second and fourth order of the gauge breaking parameter v. Our main concern is the fermionic zero modes, which are determined by iteration in v. The supersymmetric zero mode turns out to be well behaved to all orders in v, but the superconformal and quark zero modes, which conspire to develop a nonzero eigenvalue for $v \neq 0$, show nonpermissible long-distance behaviour at first order in v. The investigation is extended to $N = 2$ supersymmetric Yang-Mills theory in a separate publication [4].

Sec.II gives the general setting, with details on the action of supersymmetric QCD and its extension to Euclidean space. In sec.III the argument regarding mass corrections of the instanton is recapitulated from [2] (with an improved argument for the summation of nextleading terms at long distances), while sec.IV contains the analysis of the fermionic zero modes. The results are summarized in the conclusion, and two appendices contain some technicalities.

2 Supersymmetric QCD

2.1 The action

The action of SQCD with gauge group $SU(2)$ is:

$$S_{SQCD} = S_{\text{gauge}} + S_{\text{matter}} + \tilde{S}_{\text{matter}}. \tag{2.1}$$

Here the gauge field action is

$$S_{\text{gauge}} = \int d^4x(-i(\lambda_R^a)^\dagger(\sigma^\mu D_\mu)^{ab}\lambda_R^b - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2}D^a D^a) \tag{2.2}$$

involving the gluon field A_μ^a with field strength $F_{\mu\nu}^a$, the gluino field λ_R^a and the auxiliary field D^a, with the covariant derivative $D_\mu = \partial_\mu - ig[A_\mu, \cdot]$. The matter field action originating from one chiral superfield

$$\begin{aligned} S_{\mathbf{matter}} = \int d^4x(&-(D^\mu A)^\dagger D_\mu A - iq_L^\dagger\bar{\sigma}^\mu D_\mu q_L + F^\dagger F \\ &-ig\sqrt{2}A^\dagger\lambda_R^\dagger q_L + ig\sqrt{2}q_L^\dagger\lambda_R A - gA^\dagger D A) \end{aligned} \tag{2.3}$$

involves a complex scalar field A and a quark field q_L, both in the fundamental representation of the gauge group, with $D_\mu = \partial_\mu - igA_\mu$. $\tilde{S}_{\rm matter}$, originating from the second chiral superfield, has the same appearance as (2.3), but with a new set of "tilded" matter fields entering according to

$$q_L \rightarrow \tilde{q}_L \tag{2.4}$$

etc. Weyl spinors are used throughout, where with the conventions of Wess and Bagger [5] the metric is $\eta^{\mu\nu} = (-1, 1, 1, 1)$, while $\sigma^\mu = (-1, \vec{\sigma})$, $\bar{\sigma}^\mu = (-1, -\vec{\sigma})$, with $\vec{\sigma}$ the Pauli matrices. Isospin Pauli matrices are denoted $\vec{\tau}$. In (2.2) and (2.3)

$$A_\mu = A^a_\mu \frac{\tau^a}{2} \tag{2.5}$$

etc. The action is stationary along the flat directions where

$$D = 0,\ A = i\tau^2 \tilde{A}^* \neq 0. \tag{2.6}$$

2.2 Vainshtein-Zakharov doubling

In order to continue Majorana spinors to Euclidean space one has to double the number of components. This is conveniently done by the method of Vainshtein and Zakharov [6]. In the path integral one defines:

$$\Phi(A^a_\mu, A, \tilde{A}) = \int D\lambda^a_R D(\lambda^a_R)^\dagger Dq_L Dq^\dagger_L D\tilde{q}_L D\tilde{q}^\dagger_L$$
$$\exp(iS_{\rm SQCD,Fermi,I}) \tag{2.7}$$

where the subscript indicates that only the terms of $S_{\rm SQCD}$ involving fermions are kept. Here new integration variables are introduced in accordance with the Majorana condition for the gluino field:

$$\lambda^a_L = -i\sigma^2(\lambda^a_R)^*, \tag{2.8}$$

while for the the quark fields:

$$q_R = \tau^2 \otimes \sigma^2 q^*_L,\ \tilde{q}_R = \tau^2 \otimes \sigma^2 (\tilde{q}_L)^*. \tag{2.9}$$

Expressed in terms of the new integration variables, $\Phi(A^a_\mu, A, \tilde{A})$ becomes:

$$\Phi(A^a_\mu, A, \tilde{A}) = \int D\lambda^a_L D(\lambda^a_L)^\dagger Dq_R Dq^\dagger_R D\tilde{q}_R D\tilde{q}^\dagger_R$$
$$\exp(iS_{SQCD,\rm Fermi,II}) \tag{2.10}$$

with

$$S_{SQCD,\text{Fermi,II}} = \int d^4x(-i(\lambda_L^a)^\dagger(\bar{\sigma}^\mu D_\mu)^{ab}\lambda_L^b$$
$$-iq_R^\dagger\sigma^\mu D_\mu q_R - i\tilde{q}_R^\dagger\sigma^\mu D_\mu\tilde{q}_R$$
$$-ig\sqrt{2}(\lambda_L^a)^\dagger A^T i\tau^2\frac{\tau^a}{2}q_R - ig\sqrt{2}q_R^\dagger\frac{\tau^a}{2}i\tau^2 A^*\lambda_L^a$$
$$-ig\sqrt{2}(\lambda_L^a)^\dagger\tilde{A}^T i\tau^2\frac{\tau^a}{2}\tilde{q}_R - ig\sqrt{2}\tilde{q}_R^\dagger\frac{\tau^a}{2}i\tau^2\tilde{A}^*\lambda_L^a). \tag{2.11}$$

From (2.7) and (2.10) is formed

$$\Phi^2(A_\mu^a, A, \tilde{A}) = \int D\lambda_L^a D(\lambda_L^a)^\dagger D\lambda_R^a D(\lambda_R^a)^\dagger Dq_R Dq_R^\dagger$$
$$Dq_L Dq_L^\dagger D\tilde{q}_L D\tilde{q}_L^\dagger D\tilde{q}_R D\tilde{q}_R^\dagger$$
$$\exp(iS_{SQCD,\text{Fermi}}). \tag{2.12}$$

Here all Fermion field integration variables are independent, and

$$S_{SQCD,\text{Fermi}} = S_{SQCD,\text{Fermi,I}} + S_{SQCD,\text{Fermi,II}}. \tag{2.13}$$

This action allows continuation to Euclidean space. This is done by the following replacements:

$$x^0 \rightarrow -ix_4 \tag{2.14}$$

and [7]:

$$\lambda_R \rightarrow \lambda_A,\ \lambda_L \rightarrow \lambda_B,\ \lambda_R^\dagger \rightarrow \lambda_B^\dagger,\ \lambda_L^\dagger \rightarrow \lambda_A^\dagger \tag{2.15}$$

where the two different Euclidean space Weyl spinors are labeled A and B, with the same prescription for the quark fields. The result is:

$$\Phi^2(A_\mu^a, A, \tilde{A}) = \int D\lambda_B^a D(\lambda_A^a)^\dagger D\lambda_A^a D(\lambda_B^a)^\dagger Dq_B Dq_A^\dagger$$
$$Dq_A Dq_B^\dagger D\tilde{q}_B D\tilde{q}_A^\dagger D\tilde{q}_A D\tilde{q}_B^\dagger$$
$$\exp(\int d^4x L_{SQCD,\text{Fermi,Euclid}}) \tag{2.16}$$

with

$$L_{SQCD,\text{Fermi, Euclid}} = -(\lambda_A^a)^\dagger(\sigma\cdot D)^{ab}\lambda_B^b - (\lambda_B^a)^\dagger(\bar{\sigma}\cdot D)^{ab}\lambda_A^b$$
$$-q_A^\dagger\sigma\cdot Dq_B - q_B^\dagger\bar{\sigma}\cdot Dq_A - \tilde{q}_A^\dagger\sigma\cdot D\tilde{q}_B - \tilde{q}_B^\dagger\bar{\sigma}\cdot D\tilde{q}_A$$

$$
\begin{aligned}
&-ig\sqrt{2}A^\dagger(\lambda^a_B)^\dagger\frac{\tau^a}{2}q_B + ig\sqrt{2}q^\dagger_A\frac{\tau^a}{2}\lambda^a_A A\\
&-ig\sqrt{2}\tilde{A}^\dagger(\lambda^a_B)^\dagger\frac{\tau^a}{2}\tilde{q}_B + ig\sqrt{2}\tilde{q}^\dagger_A\frac{\tau^a}{2}\lambda^a_A \tilde{A}\\
&-ig\sqrt{2}(\lambda^a_A)^\dagger A^T i\tau^2\frac{\tau^a}{2}q_A - ig\sqrt{2}q^\dagger_B\frac{\tau^a}{2}i\tau^2 A^*\lambda^a_B\\
&-ig\sqrt{2}(\lambda^a_A)^\dagger \tilde{A}^T i\tau^2\frac{\tau^a}{2}\tilde{q}_A - ig\sqrt{2}\tilde{q}^\dagger_B\frac{\tau^a}{2}i\tau^2 \tilde{A}^*\lambda^a_B
\end{aligned}
\tag{2.17}
$$

where now $\bar{\sigma}_\mu = (i\vec{\sigma}, 1),\ \sigma_\mu = (-i\vec{\sigma}, 1)$.

3 Bosonic field equations

3.1 General Setup

For the moment fermions are ignored. The field equations of the gauge field A^a_μ and the scalar fields A and $\tilde{A}$ are from (2.2):

$$
D_\mu F^a_{\mu\nu} - igA^\dagger\frac{\tau^a}{2}\overleftrightarrow{D_\nu} A - ig\tilde{A}^\dagger\frac{\tau^a}{2}\overleftrightarrow{D_\nu}\tilde{A} = 0 \tag{3.1}
$$

and

$$
D^2 A = D^2\tilde{A} = 0. \tag{3.2}
$$

Equations (3.1) and (3.2) are simpler than the corresponding equations of the Yang-Mills-Higgs system, but a similar modification by means of a constraint is necessary [2]. It is indicated below in some detail how this comes about.

Extremizing the action with respect to the auxiliary field D one obtains the solution (2.6). The following Ansatz is used for the scalar field A:

$$
A = A(t)u_\tau;\ t = \frac{\rho^2}{x^2} \tag{3.3}
$$

with ρ the scale of the instanton and u_τ a constant unit isospinor. The gauge field is in the singular gauge supposed to have the form:

$$
A^a_\mu = -\frac{1}{g}\bar{\eta}^a_{\mu\nu}\partial_\nu \log\alpha(t), \tag{3.4}
$$

with $\bar{\eta}^a_{\mu\nu}$ the standard 't Hooft symbol. Then (3.1) and (3.2) are, expressed in terms of the functions $A(t)$ and $\alpha(t)$:

$$\frac{d}{dt}\left(\alpha^{-3}t^3\frac{d^2\alpha}{dt^2}\right) = \frac{\rho^2 g^2 A^2}{4}\alpha^{-3}\frac{d\alpha}{dt} \tag{3.5}$$

and

$$\alpha^2\frac{d^2 A}{dt^2} - \frac{3}{4}\left(\frac{d\alpha}{dt}\right)^2 A = 0. \tag{3.6}$$

3.2 Iteration

The equations (3.5) and (3.6) are solved iteratively. In the two lowest orders

$$\alpha_0 = 1 + t;\ A_1 = \frac{v}{\sqrt{1+t}} \tag{3.7}$$

where the constant v, in terms of which the expansion is carried out, is undetermined, and the subscript here and henceforth denotes the order of v.

Finiteness of the action leads according to the analysis of [2] to restrictions of the form of the prepotential α both at small and large values of t. At small values of t the leading terms of α should conspire to the modified Bessel funtion K_1:

$$\alpha = \alpha_0 + \alpha_2 + \alpha_4 + \cdots \simeq 1 + \rho g v\sqrt{t}K_1(\frac{\rho g v}{\sqrt{t}}), \tag{3.8}$$

thus ensuring exponential falloff of the gauge field. For large values of t finiteness of the action requires α to grow at most like $\log t$ to all orders in v. The properties of modified Bessel functions relevant for the present investigation are listed in Appendix B and from (B.4) follows

$$\alpha_2 \simeq \frac{\rho^2 g^2 v^2}{4}(\log\frac{\rho^2 g^2 v^2}{4t} + 2\gamma - 1) \tag{3.9}$$

valid near $t = 0$.

The equation determining α_2 is according to (3.5):

$$\frac{d}{dt}\left(\frac{t^3}{(1+t)^3}\frac{d^2\alpha_2}{dt^2}\right) = \frac{\rho^2 g^2 v^2}{4}\frac{1}{(1+t)^4}. \tag{3.10}$$

This equation has at $t \to 0$ ($x \to \infty$) the form

$$\frac{d}{dt} t^3 \frac{d^2\alpha_2}{dt^2} \simeq \frac{\rho^2 g^2 v^2}{4} \tag{3.11}$$

of which (3.9) is a solution, and for $t \to \infty$ ($x \to 0$) the form of (3.10) is:

$$\frac{d^3\alpha_2}{dt^3} \simeq O(t^{-4}) \tag{3.12}$$

that is consistent with α_2 being bounded in this limit. Thus on this level of analysis a constraint is not required.

Solving (3.10) one gets:

$$\frac{d^2\alpha_2}{dt^2} = -\frac{\rho^2 g^2 v^2}{12 t^3} + c_{2;1}(\frac{1+t}{t})^3 \tag{3.13}$$

with $c_{2;1}$ a constant of integration. The most singular part of α_2 for $t \to 0$ is then

$$\alpha_2 \simeq -\frac{1}{2t}(\frac{\rho^2 g^2 v^2}{12} - c_{2;1}). \tag{3.14}$$

Such a term is not permitted, so the value of $c_{2;1}$ is fixed:

$$c_{2;1} = \frac{\rho^2 g^2 v^2}{12}. \tag{3.15}$$

This, however, upsets the asymptotic estimate (3.12); it is replaced by:

$$\frac{d^2\alpha_2}{dt^2} \simeq c_{2;1}(\frac{3}{t} + 1) \tag{3.16}$$

leading to the solution

$$\alpha_2 = \frac{\rho^2 g^2 v^2}{4}(-\log t + t\log t - t + \frac{1}{6}t^2) + c_{2;2}t + c_{2;3} \tag{3.17}$$

where $c_{2;2}$ and $c_{2;3}$ are new integration constants, with $c_{2;2} = \frac{\rho^2 g^2 v^2}{4}$ while $c_{2;3}$ is fixed by comparison with (3.9).

The terms

$$\frac{\rho^2 g^2 v^2}{4}(t\log t + \frac{1}{6}t^2)$$

of (3.17) have to be eliminated for a finite-action solution. This can be accomplished by modifying (3.10) to:

$$\frac{d}{dt}\left(\frac{t^3}{(1+t)^3}\frac{d^2\alpha_2}{dt^2}\right)=\frac{\rho^2g^2v^2}{4}\frac{1}{(1+t)^4}-\frac{\rho^2g^2v^2}{2}\frac{t}{(1+t)^4} \tag{3.18}$$

where the extra term is introduced by a constraint. The solution of (3.18) is (3.9), now with equality sign.

At third order A_3 is according to (3.6) determined by:

$$(1+t)^2\frac{d^2A_3}{dt^2}-\frac{3}{4}A_3=\frac{3}{2}v\sqrt{1+t}\frac{d}{dt}\frac{\alpha_2}{1+t} \tag{3.19}$$

with the following solution, with appropriate choices of the two integration constants:

$$\begin{aligned} A_3 = & -\frac{1}{2}\frac{v}{\sqrt{1+t}}\frac{\alpha_2}{1+t} \\ & -\frac{\rho^2g^2v^2}{8}\frac{v}{\sqrt{1+t}}\left(t-(1+t)^2\log\frac{1+t}{t}\right). \end{aligned} \tag{3.20}$$

At fourth order the equation determining α_4 must also be modified by a constraint. It is according to (3.5):

$$\begin{aligned} & \frac{d}{dt}\left(\frac{t^3}{(1+t)^3}\frac{d^2\alpha_4}{dt^2}-3\alpha_2\frac{t^3}{(1+t)^4}\frac{d^2\alpha_2}{dt^2}\right) \\ & =\frac{\rho^2g^2v^2}{4}\left(\frac{d}{dt}\left(\frac{\alpha_2}{(1+t)^4}\right)+\frac{\frac{2A_3\sqrt{1+t}}{v}+\frac{\alpha_2}{1+t}}{(1+t)^4}\right). \end{aligned} \tag{3.21}$$

The first integration yields by (3.20) the result:

$$\begin{aligned} & \frac{t^3}{(1+t)^3}\frac{d^2\alpha_4}{dt^2}=3\alpha_2\frac{t^3}{(1+t)^4}\frac{d^2\alpha_2}{dt^2}+\frac{\rho^2g^2v^2}{4}\frac{\alpha_2}{(1+t)^4} \\ & -(\frac{\rho^2g^2v^2}{4})^2\left(-\frac{t}{1+t}\log(1+\frac{1}{t})+\frac{1}{1+t}-\frac{1}{2}\frac{1}{(1+t)^2}\right. \\ & \left.+\frac{1}{3}\frac{1}{(1+t)^3}\right)+c_{4;1} \end{aligned} \tag{3.22}$$

with $c_{4;1}$ a new integration constant. From (3.22) one finds the most singular terms of α_4 for $t \to 0$:

$$\alpha_4 \simeq (\frac{\rho^2 g^2 v^2}{4})^2 (\log \frac{\rho^2 g^2 v^2}{4t} + 2\gamma - \frac{5}{2}) \frac{1}{2t}$$
$$+(c_{4;1} - \frac{5}{6}(\frac{\rho^2 g^2 v^2}{4})^2) \frac{1}{2t}. \tag{3.23}$$

This should agree with terms of order v^4 in $\rho g v \sqrt{t} K_1(\frac{\rho g v}{\sqrt{t}})$ found from (B.4), and hence $c_{4;1}$ is fixed:

$$c_{4;1} = \frac{5}{6}(\frac{\rho^2 g^2 v^2}{4})^2 \neq 0. \tag{3.24}$$

Letting next $t \to \infty$ in (3.22) and disregarding all terms on the right hand side which vanish in this limit one gets

$$\frac{d^2 \alpha_4}{dt^2} \simeq c_{4;1}(\frac{3}{t} + 1) \tag{3.25}$$

that is similar to (3.16) and gives rise to the same problem. Thus (3.21) should have an additional term on the right hand side

$$-\, c_{4;1} \frac{6t}{(1+t)^4} \tag{3.26}$$

(cf. (3.10) and (3.18)).

A complete determination of α_4 is now possible, and the result is similar to the corresponding result for the Yang-Mills-Higgs system reported in [2].

3.3 Constraint term in the gauge field equation

The required modifications of (3.10) and (3.21) are achieved by replacing (3.5) with:

$$\alpha^2 \frac{d}{dt}\left(\alpha^{-3} t^3 \frac{d^2\alpha}{dt^2}\right) + c\frac{6t}{(1+t)^2} = \frac{\rho^2 g^2 A^2}{4} \alpha^{-1} \frac{d\alpha}{dt} \tag{3.27}$$

where

$$c = c_{2;1} + c_{4;1}. \tag{3.28}$$

Thus the gauge field equation (3.1) is modified to:

$$
\begin{aligned}
&D_\mu F^a_{\mu\nu} + \frac{c}{g}\bar{\eta}^a_{\nu\lambda}x_\lambda \frac{48\rho^2}{x^2(\rho^2+x^2)^2} \\
&-igA^\dagger \frac{\tau^a}{2} \overleftrightarrow{D}_\mu A - ig\tilde{A}^\dagger \frac{\tau^a}{2} \overleftrightarrow{D}_\mu \tilde{A} = 0.
\end{aligned} \tag{3.29}
$$

The extra term in the field equation is provided by a source term in the action that in its turn is obtained from a constraint. The scalar field equation (3.2) requires no modification.

3.4 Asymptotic Estimates

The functions α_n, $n \neq 0$, and $\sqrt{t}A_n$ will for $t \to \infty$ ($x \to 0$) in each order of v diverge at most logarithmically. This follows by induction from (3.5) (supplemented with the constraint term) and (3.6) by the proof outlined in [2].

In the opposite limit, $t \to 0$, the low order calculations lead to the following estimate:

$$
\alpha_n \propto t^{1-\frac{n}{2}} \tag{3.30}
$$

which is seen from (3.5) to be consistent to all orders. The leading term is thus for $n > 4$ more singular than $\frac{1}{t}$, and will therefore not mix with the terms arising from the integration constants, which here can be taken equal to zero. No new constraint terms are therefore required.

The leading terms of α and A in this limit, obtained by summing leading terms of α and A to all orders in v, are obtained by a series expansion in ρ with x kept fixed, or alternatively by a double series expansion in ρ and $\sqrt{t}$. The order in this new expansion is indicated by a superscript. Leading terms are

$$
A^{(0)} = v,\ \alpha^{(0)} = 1 \tag{3.31}
$$

and

$$
A^{(2)}(t) = -\frac{vt}{2}, \alpha^{(2)}(t) = \rho g v\sqrt{t}K_1(\frac{\rho g v}{\sqrt{t}}) \tag{3.32}
$$

(cf. (3.8)).

The nextleading terms can as shown in [2] be summed by Green's function techniques. We give below an improved version of this argument.

(3.5) and (3.6) are in fourth order of the expansion in ρ and $\sqrt{t}$:

$$\begin{aligned}&\frac{d^2}{dt^2}\frac{d\alpha^{(4)}}{dt}+\frac{3}{t}\frac{d}{dt}\frac{d\alpha^{(4)}}{dt}-\frac{\rho^2 g^2 v^2}{4t^3}\frac{d\alpha^{(4)}}{dt}\\&=\frac{\rho^2 g^2 v}{4t^3}(3v\alpha^{(2)}+2A^{(2)})\frac{d\alpha^{(2)}}{dt}\end{aligned}\tag{3.33}$$

and

$$\frac{d^2A^{(4)}}{dt^2}=\frac{3}{4}\left(\frac{d\alpha^{(2)}}{dt}\right)^2 v.\tag{3.34}$$

The constraint gives according to (3.18) an extra term on the right hand side of (3.33):

$$-\frac{\rho^2 g^2 v^2}{2}\frac{1}{t^2}.\tag{3.35}$$

This means that on the right hand side of (3.33), when expanded in powers of v, all $O(t^{-2})$ terms cancel out, such that the lowest order term is $O(t^{-3})$.

The solution of (3.34) is found by quadrature, where from (3.7):

$$A^{(4)}(t)=\frac{3}{8}vt^2+\cdots.\tag{3.36}$$

The solution is

$$A^{(4)}(t)=\frac{3}{4}v\int_0^t dt'(t-t')\left(\frac{d\alpha^{(2)}(t')}{dt'}\right)^2\tag{3.37}$$

showing exponential falloff for $t\to 0$, in contrast to $A^{(2)}(t)$.

(3.33) is a special case of the following equation dealt with in App.B:

$$(\frac{d^2}{dt^2}+\frac{1-n}{t}\frac{d}{dt}-\frac{\rho^2 m^2}{4t^3})f(t)=J(t)\tag{3.38}$$

where n is an integer, with the general solution

$$\begin{aligned}&f(t)=-\int_{t_0}^t dt'(f_1(t)f_2(t')-f_1(t')f_2(t))W^{-1}(t')J(t')\\&+C_1f_1(t)+C_2f_2(t)\end{aligned}\tag{3.39}$$

with C_1 and C_2 integration constants. Here $f_1(t)$ and $f_2(t)$, given in (B.8), are indepent solutions of the corresponding homogeneous equation with the Wronskian

$$W(t) = f_1(t)\frac{df_2(t)}{dt} - f_2(t)\frac{df_1(t)}{dt} \propto t^{n-1}. \tag{3.40}$$

The asymptotic form of (3.39) after summation over all orders of v can be found from (B.2), and it turns out that the term of $\alpha^{(4)}$ of lowest order in v restricts the solution in such a way that exponential increase of $\alpha^{(4)}$ for $t \to 0$ is ruled out.

Inserting the lowest powers from (B.3) and (B.4) into (B.8) (with $n = -2$) one obtains the following lowest order terms in the power series expansions of the two independent solutions $f_1(t)$ and $f_2(t)$ of the homogeneous version of (3.33):

$$f_1(t) \simeq \frac{\rho^4}{8t^2};\ f_2(t) \simeq \frac{2}{g^4 v^4}. \tag{3.41}$$

Estimating the two integral terms of (3.39) for the present case near $t \simeq 0$ one finds $W^{-1}(t)J(t) = O(t^0)$, where $J(t)$ now denotes the right hand side of (3.33). Thus

$$\int_{t_0}^{t} dt' f_1(t')W^{-1}(t')J(t') = O(t^{-1}) \tag{3.42}$$

and

$$\int_{0}^{t} dt' f_2(t')W^{-1}(t')J(t') = O(t) \tag{3.43}$$

and the two integral terms of (3.39) are both $O(t^{-1})$ in this case.

The solution $\frac{d\alpha^{(4)}}{dt}$ of (3.33) is according to the general estimate $\alpha_n = O(t^{2-\frac{n}{2}})$ for nextleading terms of order t^{-1} at lowest order in v; this is also a consequence of (3.22). The only part of (3.39) that is more singular is according to the result of the last paragraph the term $C_1 f_1(t)$. Thus $C_1 = 0$ and (3.39) becomes

$$\begin{aligned} f(t) = &-f_1(t)\int_0^t dt'(t) f_2(t')W^{-1}(t')J(t') \\ &+f_2(t)\int_{t_0}^t dt' f_1(t')W^{-1}(t')J(t') + C_2 f_2(t). \end{aligned} \tag{3.44}$$

In (3.44) all terms have exponential falloff for $t \to 0$ by (B.2) and (B.11), provided the source function $J(t)$ has this property. This is not the case for the term (3.35) arising from the constraint; however, as shown in [2] an

exponential factor may be included in the constraint to ensure exponential falloff of the nextleading terms arising from the constraint.

The iteration procedure outlined here can obviously be continued to higher orders.

4 The supersymmetric zero mode

4.1 General setup

From (2.17) one obtains the following equations for the gluino zero modes:

$$(\sigma \cdot D)^{ab} \lambda_B^b + ig\sqrt{2} A^T i\tau^2 \frac{\tau^a}{2} q_A + ig\sqrt{2} \tilde{A}^T \frac{\tau^a}{2} \tilde{q}_A = 0, \tag{4.1}$$

$$\bar{\sigma} \cdot D q_A + ig\sqrt{2} \lambda_B^a \otimes \frac{\tau^a}{2} i\tau^2 A^* = 0, \tag{4.2}$$

and

$$\bar{\sigma} \cdot D \tilde{q}_A + ig\sqrt{2} \lambda_B^a \otimes \frac{\tau^a}{2} i\tau^2 \tilde{A}^* = 0. \tag{4.3}$$

In the following boson background fields are those calculated in the previous subsection. u_σ is a constant unit two-spinor, and u_τ the corresponding isospinor introduced in (3.3), and the quark components of the zero modes are written as direct products of isospinors and spinors.

With the Ansätze, relevant for the supersymmetric zero modes in the singular gauge of the instanton field:

$$\lambda_B^a(x) = f(t) \bar{\sigma} \cdot x \sigma^a \sigma \cdot x u_\sigma, \tag{4.4}$$

$$q_A(x) = i\phi(t) \tau \cdot x \bar{\tau}_\mu i\tau^2 u_\tau \otimes \sigma_\mu u_\sigma + i\psi(t) i\tau^2 u_\tau \otimes \sigma \cdot x u_\sigma \tag{4.5}$$

and

$$\tilde{q}_A(x) = i\phi(t) \tau \cdot x \bar{\tau}_\mu u_\tau \otimes \sigma_\mu u_\sigma + i\psi(t) u_\tau \otimes \sigma \cdot x u_\sigma \tag{4.6}$$

the equations (4.1), (4.2) and (4.3) are reformulated by means of the identities listed in App.A into a set of coupled first order differential equations:

$$6f(t) - 2t\frac{df(t)}{dt} - 4t\alpha(t)^{-1}\frac{d\alpha(t)}{dt} f(t) = -g\sqrt{2} A(t) \phi(t), \tag{4.7}$$

$$4(\phi(t)+\psi(t))-2t\frac{d\psi(t)}{dt}+t\alpha^{-1}(t)\frac{d\alpha(t)}{dt}(-4\phi(t)+\psi(t))$$
$$=\frac{g}{\sqrt{2}}\frac{\rho^2}{t}A(t)f(t) \qquad (4.8)$$

and

$$2\frac{d\phi(t)}{dt}+\alpha^{-1}(t)\frac{d\alpha(t)}{dt}(-\phi(t)+\psi(t))=\frac{g}{\sqrt{2}}\frac{\rho^2}{t^2}A(t)f(t). \qquad (4.9)$$

4.2 Iteration: first four orders

The set of coupled equations (4.7), (4.8) and (4.9) is solved by iteration in the parameter v, with even orders for the function f and odd orders for the functions ϕ and ψ. The order of v is as in the previous section indicated by a subscript.

To zeroth order (4.7) implies:

$$6f_0(t)-2t\frac{df_0(t)}{dt}-\frac{4t}{1+t}f_0(t)=0 \qquad (4.10)$$

i.e.

$$f_0(t)=\frac{t^3}{(1+t)^2} \qquad (4.11)$$

where a proportionality factor was set equal to unity.

At first order (4.8) and (4.9) reduce to:

$$4(\phi_1(t)+\psi_1(t))-2t\frac{d\psi_1(t)}{dt}+\frac{t}{1+t}(-4\phi_1(t)+\psi_1(t))$$
$$=\frac{\rho^2 gv}{\sqrt{2}}\frac{t^2}{(1+t)^{\frac{5}{2}}} \qquad (4.12)$$

and

$$2\frac{d\phi_1(t)}{dt}+\frac{1}{1+t}(-\phi_1(t)+\psi_1(t))=\frac{\rho^2 gv}{\sqrt{2}}\frac{t}{(1+t)^{\frac{5}{2}}} \qquad (4.13)$$

with the solution

$$\phi_1(t)=\frac{\rho^2 gv\sqrt{2}}{8}\frac{t^2}{(1+t)^{\frac{3}{2}}},\psi_1(t)=0. \qquad (4.14)$$

At second order the following equation arises from (4.7):

$$6f_2(t) - 2t\frac{df_2(t)}{dt} - \frac{4t}{1+t}f_2(t) = -\frac{\rho^2 g^2 v^2}{4}\frac{t^2}{(1+t)^2}$$
$$+\frac{4t^4}{(1+t)^2}\left(\frac{d}{dt}\frac{\alpha_2(t)}{1+t}\right) \tag{4.15}$$

with the solution

$$f_2(t) = -\frac{\rho^2 g^2 v^2}{8}\frac{t^2}{(1+t)^2} - \frac{2t^3\alpha_2(t)}{(1+t)^3}. \tag{4.16}$$

At third order the functions $\phi_3(t)$ and $\psi_3(t)$ are determined through the coupled equations:

$$\frac{4\phi_3(t)}{(1+t)^{\frac{1}{2}}} - 2t^3(1+t)\frac{d}{dt}\frac{\psi_3(t)}{t^2(1+t)^{\frac{1}{2}}}$$
$$= gv\sqrt{2}\frac{\rho^4 g^2 v^2}{16}\left(t^2\log\frac{1+t}{t} - t\right)$$
$$-\frac{\rho^2 gv\sqrt{2}}{4}\left(\frac{5t^2}{(1+t)^3} + \frac{2t^3}{(1+t)^3}\right)\alpha_2(t) \tag{4.17}$$

and

$$2(1+t)\frac{d}{dt}\frac{\phi_3(t)}{(1+t)^{\frac{1}{2}}} + \frac{\psi_3(t)}{(1+t)^{\frac{1}{2}}}$$
$$= gv\sqrt{2}\frac{\rho^4 g^2 v^2}{16}\left(t\log\frac{1+t}{t} - 1\right)$$
$$-\frac{\rho^2 gv\sqrt{2}}{4}\left(\frac{5t}{(1+t)^3} + \frac{1}{2}\frac{t^2}{(1+t)^3}\right)\alpha_2(t)$$
$$+3gv\sqrt{2}\frac{\rho^4 g^2 v^2}{32}\frac{t}{(1+t)^2}. \tag{4.18}$$

In contrast to first order, both functions ϕ_3 and ψ_3 are nonvanishing; after some computation the following expressions are found:

$$\frac{\phi_3(t)}{(1+t)^{\frac{1}{2}}} = -\frac{\rho^2 gv\sqrt{2}}{16}\left(\frac{2t^3}{(1+t)^3} + \frac{5t^2}{(1+t)^3}\right)\alpha_2(t)$$
$$-gv\sqrt{2}\frac{\rho^4 g^2 v^2}{48}\left(\left(1 + \frac{t}{2(1+t)}\right)\log(1+t) - \frac{5}{2}t\right) \tag{4.19}$$

and

$$
\frac{\psi_3(t)}{(1+t)^{\frac{1}{2}}} = -gv\sqrt{2}\frac{\rho^4 g^2 v^2}{16}(t\log\frac{1+t}{t} - 1)
$$

$$
\left(\frac{1}{2}t + \frac{1}{6}\frac{t^2}{1+t}\right) + \frac{gv\sqrt{2}}{1+t}\frac{\rho^4 g^2 v^2}{48}(\log(1+t) - t). \tag{4.20}
$$

For $t \to \infty$ the function $\frac{\phi_3(t)}{(1+t)^{\frac{1}{2}}}$ and $\psi_3(t)(1+t)^{\frac{1}{2}}$ only have logarithmic growth. Thus the zero mode remains square integrable in this order, and no constraint is required to ensure acceptable asymptotic behaviour.

4.3 Ultraviolet asymptotic behaviour of the supersymmetric zero mode

In a similar way as in the bosonic case, it is proved by induction on the basis of (4.7), (4.8) and (4.9) that $\frac{\phi(t)}{\sqrt{t}}$, $\sqrt{t}\psi(t)$ and $f(t)$ at most have logarithmic growth for $t \to \infty$ at each order in v, except f_0 which grows linearly. Assuming that the above-mentioned estimates hold to orders less than n, one sees immediately from (4.7) that f_n is $O(t^0)$ for $t \to \infty$. From (4.8) one gets:

$$
\frac{4\phi_n(t)}{1+t} + 4\psi_n(t) - 2t\frac{d\psi_n(t)}{dt} + \frac{t\psi_n(t)}{1+t} \propto t^{-\frac{1}{2}} \tag{4.21}
$$

and from (4.9):

$$
2\frac{d\phi_n(t)}{dt} - \frac{\phi_n(t)}{1+t} + \frac{\psi_n(t)}{1+t} \propto t^{-\frac{3}{2}} \tag{4.22}
$$

establishing the estimates to order n also for the functions ϕ and ψ. Thus the estimate holds to all orders.

4.4 Infrared asymptotic behaviour of the supersymmetric zero mode

A power counting argument is carried out to determine the leading terms for $t \to 0$ of f, ϕ and ψ in each order of v. From the leading order results, extracted from (4.11), (4.14), (4.16), (4.19) and (4.20), one is lead to the hypothesis

$$
f_n \propto t^{3-\frac{n}{2}},\ \phi_n \propto t^{\frac{5}{2}-\frac{n}{2}} \tag{4.23}
$$

while $\phi_n + \psi_n$ is subleading, except at $n = 1$.

The leading terms summed to all orders in v are again obtained by a systematic double power expansion in ρ and $\sqrt{t}$, where the leading order estimates of the bosonic functions listed in (3.31) and (3.32) are combined with leading terms of f and ϕ, which are of sixth order in the double expansion of ρ and $\sqrt{t}$ and according to (4.11) and (4.14) have the following lowest order terms in the expansion of v:

$$f^{(6)}(t) = t^3 + \cdots,\ \phi^{(6)}(t) = \frac{g\sqrt{2}v\rho^2}{8}t^2 + \cdots. \tag{4.24}$$

They are solutions of

$$6f^{(6)}(t) - 2t\frac{df^{(6)}}{dt} = -g\sqrt{2}v\phi^{(6)}(t) \tag{4.25}$$

and

$$\frac{2t^2}{\rho^2}\frac{d\phi^{(6)}}{dt} = \frac{gv}{\sqrt{2}}f^{(6)}(t) \tag{4.26}$$

following from (4.7), (4.8) and (4.9), where it also is used that $\phi + \psi$ only has an eighth order term. Combining (4.25) and (4.26) one gets:

$$\frac{d^2 f^{(6)}(t)}{dt^2} - \frac{2}{t}\frac{df^{(6)}(t)}{dt} - \frac{g^2v^2\rho^2}{4t^3}f^{(6)}(t) = 0 \tag{4.27}$$

and

$$\frac{d^2\phi^{(6)}(t)}{dt^2} - \frac{1}{t}\frac{d\phi^{(6)}(t)}{dt} - \frac{g^2v^2\rho^2}{4t^3}\phi^{(6)}(t) = 0 \tag{4.28}$$

which are special cases of (3.39) and have the following solutions, obtained by combination of (B.8), the lowest order terms of the modified Bessel functions (B.3) and (B.4), and finally (4.24):

$$f^{(6)}(t) = \frac{\rho^6}{8}(\frac{gv\sqrt{t}}{\rho})^3 K_3(\frac{\rho g v}{\sqrt{t}})), \tag{4.29}$$

$$\phi^{(6)}(t) = \frac{g\sqrt{2}v\rho^6}{16}(\frac{gv\sqrt{t}}{\rho})^2 K_2(\frac{\rho g v}{\sqrt{t}}). \tag{4.30}$$

It is an important point that the modified Bessel functions I_2 and I_3 do not occur here; as a consequence the two solutions show exponential decrease for $t \to 0$ ($t \to \infty$).

We show explicitly how the term added to the first logarithmic part of f and ϕ can be chosen freely as integration constants, thus excluding the presence I_2 and I_3 in the solutions. We thus return to the power series expansion in v, where logarithmic terms occur in ϕ_5 and f_6. Hence we are led to consider separately:

$$-2t^4 \frac{d}{dt}\frac{f_4(t)}{t^3} \simeq -g\sqrt{2}v\phi_3(t), \tag{4.31}$$

$$\frac{d\phi_5(t)}{dt} \simeq \frac{g\sqrt{2}v\rho^2}{4t^2} f_4(t) \tag{4.32}$$

and

$$-2t^4 \frac{d}{dt}\frac{f_6(t)}{t^3} \simeq -g\sqrt{2}v\phi_5(t) \tag{4.33}$$

with, according to (4.19):

$$\phi_3(t) \simeq -g\sqrt{2}v\frac{\rho^4 g^2 v^2}{32}t. \tag{4.34}$$

The solutions of these equations are first

$$f_4(t) \simeq \frac{\rho^4 g^4 v^4}{64}t \tag{4.35}$$

and

$$\phi_5(t) \simeq -g\sqrt{2}v\frac{\rho^4 g^4 v^4}{256}(\log\frac{\rho^2 m^2}{4t} + 2\gamma - \frac{3}{2}) \tag{4.36}$$

where an integration constant is chosen in accordance with (B.4). Finally $f_6(t)$ is determined by:

$$\frac{d}{dt}\frac{f_6(t)}{t^3} \simeq -\frac{\rho^4 g^6 v^6}{256}(\log\frac{\rho^2 m^2}{4t} + 2\gamma - \frac{3}{2})\frac{1}{t^4} \tag{4.37}$$

whence

$$f_6(t) \simeq \frac{\rho^4 g^6 v^6}{768}(\log\frac{\rho^2 m^2}{4t} + 2\gamma - \frac{3}{2} - \frac{1}{3}) \tag{4.38}$$

in exact ageement with (B.4). Thus correct Bessel function behaviour of both the quark and gluino functions is ensured once the integration constant in ϕ_5 has been chosen properly.

Nextleading terms are of eighth order in ρ and $\sqrt{t}$ and are according to (4.7), (4.8) and (4.9) determined by:

$$\frac{d^2 f^{(8)}(t)}{dt^2} - \frac{2}{t}\frac{df^{(8)}(t)}{dt} - \frac{g^2 v^2 \rho^2}{4t^3} f^{(8)}(t)$$
$$= \frac{g\sqrt{2} v \rho^2}{4t^3}\left(\frac{g}{\sqrt{2}} A^{(2)} f^{(6)}(t) + \frac{3t^2}{\rho^2}\frac{d\alpha^{(2)}(t)}{dt}\phi^{(6)}(t)\right)$$
$$-\frac{1}{2t}\frac{d}{dt}\left(-g\sqrt{2} v A^{(2)}\phi^{(6)} + 4t\frac{d\alpha^{(2)}}{dt} f^{(6)}(t)\right), \tag{4.39}$$

$$\frac{d^2 \phi^{(8)}}{dt^2} - \frac{1}{t}\frac{d\phi^{(8)}}{dt} - \frac{g^2 v^2 \rho^2}{4t^3}\phi^{(8)}(t)$$
$$= \frac{1}{t}\frac{d}{dt}\frac{\rho^2}{2t^3}\left(\frac{g}{\sqrt{2}} A^{(2)} f^{(6)}(t) + \frac{3t^2}{\rho^2}\frac{d\alpha^{(2)}(t)}{dt}\phi^{(6)}(t)\right)$$
$$+\frac{g v \rho^2}{\sqrt{2}}\frac{1}{4t^5}\left(-g\sqrt{2} A^{(2)}(t)\phi^{(6)}(t) + 4t\frac{d\alpha^{(2)}(t)}{dt} f^{(6)}(t)\right) \tag{4.40}$$

and

$$-2t\frac{d\tilde{\psi}^{(8)}(t)}{dt} + 4\tilde{\psi}^{(8)}(t) = 3t\frac{d\alpha^{(2)}(t)}{dt}\phi^{(6)}(t) \tag{4.41}$$

with $\tilde{\psi}^{(8)}(t) = \psi^{(8)}(t) + \phi^{(8)}(t)$. (4.39) and (4.40) are special cases of (3.38), with $n = 3$ and $n = 2$, respectively, and their solutions are found from (3.39), while (4.41) is solved by quadrature. All the terms on the right-hand sides of (4.39), (4.40) and (4.41) have exponential decrease for $t \to 0$. This property is, as is demonstrated below, shared by the solutions.

The solution of (4.41) is

$$\tilde{\psi}^{(8)}(t) = -\frac{3}{2}t^2\int_0^t \frac{dt'}{(t')^2}\frac{d\alpha^{(2)}(t')}{dt'}\phi^{(6)}(t') \tag{4.42}$$

where by (4.24) we have $\phi^{(6)}(t) = O(t^2)$ in lowest order of the expansion in v, and inserting lowest order terms in (4.42) we find

$$\tilde{\psi}^{(8)}(t) = -\frac{3}{16} g\sqrt{2} v \rho^2 t^3 + \cdots \tag{4.43}$$

in agreement with (4.14). The integrand of (4.42) shows exponential decrease for $t \to 0$ after summation over all orders of v, and the integral therefore also does so.

To the lowest order of v the right-hand side of (4.39) is $O(t)$ and of (4.40) it is $O(t^{-1})$ while the solutions are $O(t^4)$ and $O(t^3)$. In the integral of (3.39):

$$\int_{t_0}^{t} dt'(f_1(t)f_2(t') - f_1(t')f_2(t))W^{-1}(t')J(t') \tag{4.44}$$

the estimates $f_1(t) = O(t^0)$, $f_2(t) = O(t^n)$ following from (B.8), (B.3) and (B.4) for $n > 0$ are used. For (4.39) one finds

$$\begin{aligned} &\int_0^t dt' f_2(t')W^{-1}(t')J(t') \\ &= \int_0^t dt' O((t')^3)O((t')^{-2})O(t') = O(t^3) \end{aligned} \tag{4.45}$$

and

$$\begin{aligned} &\int_{t_0}^t dt' f_1(t')W^{-1}(t')J(t') \\ &= \int_{t_0}^t dt' O((t')^0)O((t')^{-2}), O(t') = O(t^0) \end{aligned} \tag{4.46}$$

respectively, so (4.44) is $O(t^3)$. For (4.40) the results are

$$\begin{aligned} &\int_0^t dt' f_2(t')W^{-1}(t')J(t') \\ &= \int_0^t dt' O((t')^2)O((t')^{-1})O((t')^{-1}) = O(t) \end{aligned} \tag{4.47}$$

and

$$\begin{aligned} &\int_{t_0}^t dt' f_1(t')W^{-1}(t')J(t') \\ &= \int_{t_0}^t dt' O((t')^0)O((t')^{-1})O((t')^{-1}) = O(t^{-1}) \end{aligned} \tag{4.48}$$

so here (4.44) is $O(t)$. Thus in neither case is there is any possibility of having a term proportional to $f_1(t)$ in the solution, and asymptotic falloff is ensured.

As in the bosonic case this iteration procedure can be continued and the conclusions carry over to higher orders.

5 The superconformal zero mode

The equations (4.1)-(4.3) also have as a solution the superconformal zero mode. In this case the fields have the forms:

$$\lambda^a_B = f(t)\bar{\sigma}\cdot x\sigma^a u_\sigma, \tag{5.1}$$

$$q_A = i\phi(t)\tau^a i\tau_2 u_\tau \otimes \sigma^a u_\sigma + i\psi(t) i\tau 2 u_\tau \otimes u_\sigma \tag{5.2}$$

and

$$\tilde{q}_A = i\phi(t)\tau^a u_\tau \otimes \sigma_a u_\sigma + i\psi(t) u_\tau \otimes u_\sigma. \tag{5.3}$$

Then the following set of coupled equations is obtained by the identities of App. A:

$$4f(t) - 2t\frac{df(t)}{dt} - 4t\frac{d\log\alpha(t)}{dt}f(t) = -g\sqrt{2}A(t)\phi(t), \tag{5.4}$$

$$2\frac{d\psi(t)}{dt} + 3\frac{d\log\alpha(t)}{dt}\phi(t) = 0 \tag{5.5}$$

and

$$2\frac{d\phi(t)}{dt} + \frac{d\log\alpha(t)}{dt}(\psi(t) - 2\phi(t)) = g\sqrt{2}\frac{\rho^2}{2t^2}A(t)f(t). \tag{5.6}$$

5.1 Iteration up to first order

At zeroth order the solution of (5.4) is:

$$f_0(t) = \frac{t^2}{(1+t)^2}. \tag{5.7}$$

At first order (5.5) and (5.6) are:

$$2\frac{d\psi_1(t)}{dt} + \frac{3\phi_1(t)}{1+t} = 0 \tag{5.8}$$

and

$$2\frac{d\phi_1(t)}{dt} + \frac{\psi_1(t) - 2\phi_1(t)}{1+t} = \frac{g\sqrt{2}v\rho^2}{2}\frac{1}{(1+t)^{\frac{5}{2}}} \tag{5.9}$$

with the solutions:

$$\phi_1(t) = \frac{1}{4}\frac{C_1}{(1+t)^{\frac{1}{2}}} + \frac{3}{4}C_2(1+t)^{\frac{3}{2}} - \frac{g\sqrt{2}v\rho^2}{8}\frac{1}{(1+t)^{\frac{3}{2}}} \tag{5.10}$$

and

$$\psi_1(t) = \frac{3}{4}\frac{C_1}{(1+t)^{\frac{1}{2}}} - \frac{3}{4}C_2(1+t)^{\frac{3}{2}} - \frac{g\sqrt{2}v\rho^2}{8}\frac{1}{(1+t)^{\frac{3}{2}}} \tag{5.11}$$

where C_1 and C_2 are integration constants.

A solution that vanishes at $t=0$ is obtained by the choice:

$$C_1 = \frac{g\sqrt{2}v\rho^2}{4};\ C_2 = \frac{g\sqrt{2}v\rho^2}{12} \tag{5.12}$$

i.e.

$$\phi_1(t) = \frac{g\sqrt{2}v\rho^2}{8}(\frac{1}{2}\frac{1}{(1+t)^{\frac{1}{2}}} + \frac{1}{2}(1+t)^{\frac{3}{2}} - \frac{1}{(1+t)^{\frac{3}{2}}}) \tag{5.13}$$

and

$$\psi_1(t) = \frac{g\sqrt{2}v\rho^2}{8}(\frac{3}{2}\frac{1}{(1+t)^{\frac{1}{2}}} - \frac{1}{2}(1+t)^{\frac{3}{2}} - \frac{1}{(1+t)^{\frac{3}{2}}}). \tag{5.14}$$

However, this solution is singular at the origin; a solution that is regular at the origin (for $t \to \infty$) can only be obtained for $C_2 = 0$.

It was suggested [3] that a regular superconformal zero mode can be obtained by introducing additional Yukawa coupling terms in the Lagrangian through a constraint. Several possibilities for doing so suggest themselves.

The terms that grow too fast for $t \to \infty$ in (5.13) and (5.14) are eliminated by adding to the solution:

$$\phi_{1,\text{add}}(t) = -\frac{g\sqrt{2}v\rho^2}{16}\frac{2t+t^2}{\sqrt{1+t}} \tag{5.15}$$

and

$$\psi_{1,\text{add}}(t) = \frac{g\sqrt{2}v\rho^2}{16}\frac{2t+t^2}{\sqrt{1+t}}. \tag{5.16}$$

This means replacing (4.2) by:

$$\begin{aligned}
&\bar{\sigma}\cdot Dq_A + ig\sqrt{2}\frac{\tau^a}{2}\lambda_B^a i\tau^2 A^* \\
&= i\frac{t^2}{\rho^2}\frac{g\sqrt{2}v\rho^2}{4}(1+t)^{-\frac{3}{2}}(\tau^a u_\tau \otimes \bar{\sigma}\cdot x\sigma^a u_\sigma \\
&-u_\tau \otimes \bar{\sigma}\cdot x u_\sigma).
\end{aligned} \tag{5.17}$$

In order to produce the right-hand side by a Yukawa coupling one evidently needs an isoscalar field in addition to the isovector gluino field λ_B^a. This

feature persists when in (5.15) and (5.16) the numerator $2t + t^2$ is replaced by a different function of t.

Keeping instead $C_2 = 0$ but choosing:

$$C_1 = \frac{g\sqrt{2}v\rho^2}{2} \tag{5.18}$$

one obtains that $\phi_1(t)$ vanishes for $t = 0$, and (5.10) and (5.11) are:

$$\phi_1(t) = \frac{g\sqrt{2}v\rho^2}{8}\left(\frac{1}{(1+t)^{\frac{1}{2}}} - \frac{1}{(1+t)^{\frac{3}{2}}}\right) \tag{5.19}$$

and

$$\psi_1(t) = \frac{g\sqrt{2}v\rho^2}{8}\left(\frac{3}{(1+t)^{\frac{1}{2}}} - \frac{1}{(1+t)^{\frac{3}{2}}}\right). \tag{5.20}$$

We can instead of (5.20) postulate a solution obtained by addition of

$$\psi_{1,\text{add}}(t) = -\frac{g\sqrt{2}v\rho^2}{4} \tag{5.21}$$

thus obtaining also $\psi_1(0) = 0$. This is achieved by modifying (4.2) into

$$\begin{aligned}
&\bar{\sigma} \cdot Dq_A + ig\sqrt{2}\frac{\tau^a}{2}\lambda_B^a i\tau^2 A^* \\
&= i\frac{g\sqrt{2}v}{4}\frac{t^2}{1+t}\tau^a i\tau^2 u_\tau \otimes \bar{\sigma} \cdot x\sigma^a u_\sigma \\
&\simeq i\frac{g\sqrt{2}v}{4}(1+t)\tau^a i\tau^2 u_\tau \otimes \lambda_B^a,
\end{aligned} \tag{5.22}$$

where no isoscalar term occurs, and (4.3) should be modified correspondingly. Thus the following new Yukawa coupling terms should be present in (2.17):

$$-i\frac{g\sqrt{2}v}{4}(1+t)(q_B^\dagger \tau^a i\tau^2 u_\tau + \tilde{q}_B^\dagger \tau^a u_\tau) \otimes \lambda_B^a. \tag{5.23}$$

5.2 Ultraviolet and infrared asymptotic behaviour

For $t \to \infty$ one has from (5.4), (5.5) and (5.6) and from the low order computations the following consistent estimates:

$$f_0 \to 1;\ f_n \propto t^{-1},\ n \neq 0; \phi \propto t^{-\frac{3}{2}}, \phi - \psi \propto t^{-\frac{5}{2}}. \tag{5.24}$$

For $t \to 0$ leading, nextleading etc. terms are found by a systematic expansion in ρ and $\sqrt{t}$, starting in second order, where the solutions (5.19)-(5.20) imply

$$\psi^{(2)} = \frac{g\sqrt{2}v\rho^2}{4} + \cdots \tag{5.25}$$

while in fourth order

$$\begin{aligned} &f^{(4)}(t) = t^2 + \cdots,\ \psi^{(4)}(t) = O(t^2), \\ &\phi^{(4)}(t) = \frac{g\sqrt{2}v\rho^2}{8}t + \cdots. \end{aligned} \tag{5.26}$$

The equations (5.4), (5.5) and (5.6) require in second order that $\psi^{(2)}$ is indeed a constant while the nextleading terms are analyzed in the same way and with the same conclusions as for the supersymmetric zero mode.

6 The quark zero mode

The quark zero mode is according to (2.17) determined by:

$$\sigma \cdot D q_B - ig\sqrt{2}\frac{\tau^a}{2}\lambda_A^a A = 0, \tag{6.27}$$

$$\sigma \cdot D \tilde{q}_B - ig\sqrt{2}\frac{\tau^a}{2}\lambda_A^a \tilde{A} = 0 \tag{6.28}$$

and

$$(\bar{\sigma} \cdot D)^{ab}\lambda_A^b + ig\sqrt{2}A^\dagger\frac{\tau^a}{2}q_B + ig\sqrt{2}\tilde{A}^\dagger\frac{\tau^a}{2}\tilde{q}_B = 0. \tag{6.29}$$

In (6.27)-(6.29) the following Ansätze are used:

$$q_B(x) = \phi(t)\tau^a u_\tau \otimes \bar{\sigma} \cdot x\sigma^a u_\sigma + \psi(t)u_\tau \otimes \bar{\sigma} \cdot x u_\sigma, \tag{6.30}$$

$$\begin{aligned} &\tilde{q}_B(x) = \phi(t)\tau^a(-i\tau^2)u_\tau \otimes \bar{\sigma} \cdot x\sigma^a u_\sigma \\ &+\psi(t)(-i\tau^2)u_\tau \otimes \bar{\sigma} \cdot x u_\sigma \end{aligned} \tag{6.31}$$

and

$$\lambda_A^a(x) = if(t)\sigma^a u_\sigma \tag{6.32}$$

and one obtains the three equations:

$$4\phi(t) - 2t\frac{d\phi(t)}{dt} - 2t\frac{d\log\alpha(t)}{dt}\phi(t) + t\frac{d\log\alpha(t)}{dt}\psi(t)$$
$$= -\frac{g}{\sqrt{2}}A(t)f(t), \tag{6.33}$$

$$4\psi(t) - 2t\frac{d\psi(t)}{dt} + 3t\frac{d\log\alpha(t)}{dt}\phi(t) = 0 \tag{6.34}$$

and

$$-\frac{2t^2}{\rho^2}\left(\frac{df(t)}{dt} - 2\frac{d\log\alpha(t)}{dt}f(t)\right) = -g\sqrt{2}A(t)\phi(t). \tag{6.35}$$

To zeroth order the solutions are

$$\phi_0(t) = -\psi_0(t) = \frac{t^2}{(1+t)^{\frac{3}{2}}} \tag{6.36}$$

whence the following first order solution of (6.35) is found

$$f_1(t) = -\frac{g\sqrt{2}v\rho^2}{6}\frac{1}{1+t}. \tag{6.37}$$

Adding to $f(t)$:

$$f_{1,\mathrm{add}}(t) = \frac{g\sqrt{2}v\rho^2}{6} \tag{6.38}$$

one obtains $f(0) = 0$. This is achieved by the new Yukawa coupling terms in (2.17):

$$-\frac{2}{3}igv\sqrt{2}(1+t)^{\frac{1}{2}}(\lambda_B^a)^\dagger(x)(u_\tau^\dagger\frac{\tau^a}{2}q_B(x)$$
$$+u_\tau^\dagger i\tau^2\frac{\tau^a}{2}\tilde{q}_B(x)). \tag{6.39}$$

They are similar to (5.23) but with a different coefficient. Considerations on matters of asymptotic behaviour are very similar both in methods and results to the analysis presented above for the superconformal zero mode.

7 Conclusion

In this paper a systematic way of dealing with mass corrections of fermionic zero modes in the presence of constrained instantons was presented, following the general pattern applied to the constrained instanton itself in [2]. Low order iterations as well as general estimates of the asymptotic behaviour both at long and short distances were presented. While the supersymmetric zero mode was found to be well behaved, the superconformal and quark zero modes go at large distances as constants at first order in v.

However it was found, following the suggestion of [3], that these unpermissible first order terms can be removed by new Yukawa couplings, which consequently are of first order in v, in contrast to the gauge constraint terms, which are of second and fourth order in v.

No operator constraint was determined. As pointed out in [2], the existence of a gauge invariant operator constraint is unlikely, and even if it exists, it can probably not be implemented in a supersymmetric fashion because of the mismatch of the powers of v in the bosonic and the fermionic sector.

A Algebraic identities

A number of useful algebraic identities involving the matrices $\sigma_\mu = (-i\vec{\sigma}, 1)$ and $\bar{\sigma}_\mu = (i\vec{\sigma}, 1)$ is recorded:

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = \bar{\sigma}_\mu \sigma_\nu + \bar{\sigma}_\nu \sigma_\mu = 2\delta_{\mu\nu}, \tag{A.1}$$

$$\bar{\eta}^a_{\mu\nu} \sigma_\mu x_\nu = i\sigma^a \sigma \cdot x;\ \bar{\eta}^a_{\mu\nu} \bar{\sigma}_\mu x_\nu = -i\bar{\sigma} \cdot x \sigma^a, \tag{A.2}$$

$$\bar{\sigma}_\mu \sigma^a \sigma_\mu = 0, \tag{A.3}$$

$$\tau_\lambda \bar{\tau}_\mu u_\tau \otimes \bar{\sigma}_\lambda \sigma_\mu u_\sigma = 4 u_\tau \otimes u_\sigma, \tag{A.4}$$

$$\begin{aligned} &\tau \cdot x \bar{\tau}_\lambda u_\tau \otimes \bar{\sigma} \cdot x \sigma_\lambda u_\sigma \\ &= x^2 u_\tau \otimes u_\sigma + \tau^a u_\tau \otimes \bar{\sigma} \cdot x \sigma^a \sigma \cdot x u_\sigma \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} &i\bar{\eta}^a_{\mu\nu} x_\nu \tau^a \tau \cdot x \bar{\tau}_\lambda u_\tau \otimes \bar{\sigma}_\mu \sigma_\lambda u_\sigma \\ &= 4x^2 u_\tau \otimes u_\sigma - \tau \cdot x \bar{\tau}_\lambda u_\tau \otimes \bar{\sigma} \cdot x \sigma_\lambda u_\sigma. \end{aligned} \tag{A.6}$$

B Bessel functions

The modified Bessel equation [8] of order n

$$z^2\frac{d^2G_n(z)}{dz^2}+z\frac{dG_n(z)}{dz}-(z^2+n^2)G_n(z)=0 \tag{B.1}$$

has two linearly independent solutions $I_n(z)$ and $K_n(z)$. For $z\to\infty$:

$$I_n(z)\simeq\sqrt{\frac{1}{2\pi x}}e^z;\ K_n(z)\simeq\sqrt{\frac{\pi}{x}}e^{-z}. \tag{B.2}$$

The power series expansions of the modified Bessel functions are

$$I_n(z)=(\frac{z}{2})^n\sum_{r=0}^{\infty}\frac{(\frac{z^2}{4})^r}{r!(n+r)!} \tag{B.3}$$

and

$$\begin{aligned}&K_n(z)=(-1)^{n+1}I_n(z)(\log\frac{z}{2}+\gamma)\\&+\frac{1}{2}\sum_{r=0}^{n-1}\frac{(-1)^r(n-r-1)!}{r!}(\frac{z}{2})^{-n+2r}\\&+\frac{1}{2}(-\frac{z}{2})^n\sum_{r=0}^{\infty}\frac{1}{r!(n+r)!}(\phi(r)+\phi(n+r))(\frac{z^2}{4})^r\end{aligned} \tag{B.4}$$

with

$$\phi(0)=0,\ \phi(r)=\sum_{s=1}^{r}\frac{1}{s}, r\neq 0. \tag{B.5}$$

From the definition $t=\frac{\rho^2}{x^2}$ follows:

$$\begin{aligned}&(\frac{d^2}{dt^2}+\frac{1-n}{t}\frac{d}{dt}-\frac{\rho^2m^2}{4t^3})(\frac{m}{x})^nG_n(mx)\\&=\frac{1}{x^2}\frac{\rho^2}{4t^3}(\frac{m}{x})^n[m^2x^2G_n''(mx)+mxG_n'(mx)\\&-(m^2x^2+n^2)G(mx)].\end{aligned} \tag{B.6}$$

Here the expression in the square bracket is recognized as the defining equation of the modified Bessel functions (B.8) of order n. Consequently the solutions of the equation

$$(\frac{d^2}{dt^2}+\frac{1-n}{t}\frac{d}{dt}-\frac{\rho^2m^2}{4t^3})f(t)=0 \tag{B.7}$$

are

$$f_1(t) = (\frac{m}{x})^n I_n(mx) \text{ or } f_2(t) = (\frac{m}{x})^n K_n(mx). \tag{B.8}$$

In terms of t and using the estimates (B.2) valid near $t = 0$ the Wronskian $W(t)$ of the two solutions is:

$$W(t) = f_1(t)\frac{df_2(t)}{dt} - f_2(t)\frac{df_1(t)}{dt} \simeq \frac{1}{\sqrt{2}}(\frac{m}{\rho})^{2n} t^{n-1}. \tag{B.9}$$

It fulfils:

$$\frac{dW(t)}{dt} = \frac{n-1}{t} W(t). \tag{B.10}$$

Thus (B.9), which was derived only for $t \simeq 0$, actually is valid as an equality for all values of t.

The general inhomogeneous version of (B.7) is (3.38) with the general solution (3.39). The integral here is near $t = 0$ by (B.2) and (B.9) proportional to

$$-2\int^t dt'(\sqrt{tt'})^n \frac{\sqrt[4]{tt'}}{m\rho}(t')^{1-n}\sinh\rho(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t'}})J(t'). \tag{B.11}$$

If $J(t)$ falls off exponentially for $t \to 0$ it is immediately clear from (3.39) or (B.11) that the same holds for $f(t)$.

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